Riemannian Geometry $\boxed{\text{Week 11}}$

Last time: If $N \subseteq (M,g)$ is a submanifold:

$\Gamma(M,N)$: Vector fields on M tangent to N.

Weingarten equation: If $X, Y \in \Gamma(M,N)$, $W \in \Gamma(M)$, $W|_p \perp T_pN$ $\forall p \in N$, then

$$\langle D_X W, Y\rangle = -\langle W, \underbrace{B(X,Y)}_{= (D_X Y)^\perp}\rangle$$

and $(\bar{D}_X Y) = (D_X Y)^T$

and:

Gauss equation:

$\forall\ X, Y, Z, W \in T_pN$:

$$\bar{R}(X,Y,Z,W) - R(X,Y,Z,W) = \langle B(X,Z), B(Y,W)\rangle - \langle B(X,W), B(Y,Z)\rangle$$

ass. to $\bar{g}$ (induced metric)

Proof: I can extend X, Y, Z, W to $\Gamma(M,N)$.

Without loss of generality: $[X,Y]|_p = 0$.

Then:

(recall: $R(X,Y,Z,W) = \langle R(X,Y)W, Z\rangle$

$$\bar{D}_X \bar{D}_Y W = D_X(\bar{D}_Y W) - B(X, \bar{D}_Y W)$$

$$= D_X(D_Y W) - D_X(B(Y,W)) - B(X, \bar{D}_Y W)$$

$B \perp N$

So: $\langle \bar{D}_X \bar{D}_Y W, Z\rangle = \langle D_X D_Y W, Z\rangle - \langle D_X(B(Y,W)), Z\rangle - \underbrace{\cancel{\langle B(X, \bar{D}_Y W), Z\rangle}}_{0}$

$$= \langle D_X D_Y W, Z\rangle \overset{\text{Weingarten}}{+} \langle B(Y,W), D_X Z\rangle$$

$$= \langle D_X D_Y W, Z\rangle + \langle B(Y,W), B(X,Z) + \underbrace{\cancel{\text{tangent}}}_{0}\rangle$$

Subtracting the same relation with $X \leftrightarrow Y$, I get the Gauss equation $\square$

When N is a <u>hypersurface</u> (i.e. codim 1), <u>co-oriented</u> (with unit normal $\hat{n}$):

Scalar second fundamental form:

$$b(X,Y) := \langle B(X,Y), \hat{n} \rangle$$

Weingarten map: $X \in T_pN \longrightarrow \overbrace{(\nabla_X \hat{n})^T}^{=: L(X)} \in T_pN$

So $b(X,Y) = \langle \nabla_X Y, \hat{n} \rangle = \langle -\nabla_X \hat{n}, Y \rangle = -\langle L(X), Y \rangle$

Symmetry of $b \Rightarrow \langle L(X), Y \rangle = \langle L(Y), X \rangle$

<u>Principal curvatures</u> at a point p: $k_1(p), \dots, k_n(p)$:

Eigenvalues of L_p (real)

<u>Gauss curvature</u>:

$$K_{Gauss} = \det L = k_1 \cdots k_n$$

<u>Mean curvature</u>: $H(p) := \frac{1}{n} \operatorname{tr} L = \frac{k_1 + \dots + k_n}{n}$

<u>Umbilic point</u>: $k_1 = k_2 = \dots = k_n$

If $N^2 \subset M^3$: Gauss equation: $\overline{K}(p) = K(p) + K_{Gauss}(p)$

In $\mathbb{R}^3$: ($\overline{K}(p) = 0$) Gauss curvature is <u>intrinsic</u> ($= \frac{1}{2} S$)

If the hypersurface $N \subset M$ is given as the level set $\{f = \text{const}\}$ with $df|_p \neq 0$:

$$\hat{n} = \frac{\operatorname{grad} f}{\|\operatorname{grad} f\|} \qquad (\operatorname{grad} f)^a = g^{ab} \partial_b f$$

$$b(X,Y) = -\frac{\operatorname{Hess} f\,(X,Y)}{\|\operatorname{grad} f\|},$$

$$\operatorname{Hess} f = \nabla df \quad \text{so} \quad (\operatorname{Hess} f)_{ab} = \partial^2_{ab} f - \Gamma^k_{ab} \partial_k f$$

In general: $\boxed{b = -\frac{1}{2} \mathcal{L}_{\hat{n}} g}$

Second variation formula:

We have seen that curves which ~~[illegible]~~ are stationary points of the length functional are geodesics. When are geodesics (locally) length minimizing?

~~[illegible]~~ The answer to this question has topological consequences

Theorem (2nd variation formula)

Let $\gamma : [a,b] \to (M,g)$ be a geodesic parametrized by arc length ($\|\dot{\gamma}\| = 1$). Let $\phi : (-\varepsilon,\varepsilon) \times [a,b] \to M$ be a variation of γ ($\phi_0(t) = \gamma(t)$). Let $X = d\phi(\frac{\partial}{\partial s}) = \frac{\partial \phi}{\partial s}$ be the variation vector field.

Then
$$\frac{d^2}{ds^2} l(\phi_s)\Big|_{s=0} = \langle \nabla_X X, \dot{\gamma} \rangle \Big|_{t=a}^{b} + \int_a^b \Big(\|\nabla_{\dot{\gamma}} X^{\perp}\|^2 - \underbrace{R(\dot{\gamma}, X, \dot{\gamma}, X)}_{K(\dot{\gamma}, X^{\perp}) \|X^{\perp}\|^2} \Big)\, dt$$

where $Y^{\perp} = Y - \frac{\langle Y, \dot{\gamma} \rangle}{\|\dot{\gamma}\|^2} \dot{\gamma}$.

We will denote with $T = d\phi(\frac{\partial}{\partial t})$ the tangent to ϕ_s.

<u>Lem</u>: $\nabla_T X = \nabla_X T$

and $\langle \nabla_X \nabla_T X, T \rangle = \langle \nabla_T \nabla_X X, T \rangle - R(T, X, T, X)$

<u>Proof</u>: $\nabla_T X - \nabla_X T = [T, X] = [d\phi(\frac{\partial}{\partial t}), d\phi(\frac{\partial}{\partial s})] = d\phi(\overset{0}{[\cancel{\frac{\partial}{\partial t}}, \frac{\partial}{\partial s}]})$

And $R(X,T)X = \nabla_X \nabla_T X - \nabla_T \nabla_X X$

and $R(T, X, T, X) =$

$- R(X, T, T, X) = \langle R(X,T)X, T \rangle$ ∎

<u>Lem</u>: For any variation:

$$\frac{d}{ds} l(\phi_s) = \int_a^b \frac{\langle \nabla_T X, T \rangle}{\|T\|} dt \qquad \text{(not just at } s=0\text{)}$$

<u>Proof</u>: $l(\phi_s) = \int_a^b \|T\| dt = \int_a^b \sqrt{\langle T, T \rangle} dt$

$$\frac{d}{ds} l(\phi_s) = \int_a^b \frac{d}{ds} \sqrt{\langle T, T \rangle} dt = \int_a^b \frac{1}{2} \frac{\overset{X}{\frac{d}{ds}} \langle T, T \rangle}{\sqrt{\langle T, T \rangle}} dt$$

$$= \int_a^b \frac{\langle \nabla_X T, T \rangle}{\sqrt{\langle T, T \rangle}} dt \overset{[X,T]=0}{=} \int_a^b \frac{\langle \nabla_T X, T \rangle}{\|T\|} dt$$

<u>Lem</u>: At $s=0$: $\|\nabla_{\dot\gamma} X^\perp\|^2 = \|\nabla_{\dot\gamma} X\|^2 - \langle \nabla_{\dot\gamma} X, \dot\gamma \rangle^2$

<u>Proof</u>: and $(\nabla_{\dot\gamma} X)^\perp = \nabla_{\dot\gamma}(X^\perp)$

$$\nabla_{\dot\gamma} X = (\nabla_{\dot\gamma} X)^{\perp} + \langle \nabla_{\dot\gamma} X, \dot\gamma \rangle \dot\gamma$$

orthogonal

$$\Rightarrow \quad \|\nabla_{\dot\gamma} X\|^2 = \|(\nabla_{\dot\gamma} X)^{\perp}\|^2 + \langle \nabla_{\dot\gamma} X, \dot\gamma \rangle^2 .$$

And $\nabla_{\dot\gamma}(X^{\perp}) = \nabla_{\dot\gamma}(X - \langle X, \dot\gamma\rangle \dot\gamma) = \nabla_{\dot\gamma} X - \langle \nabla_{\dot\gamma} X, \dot\gamma \rangle \dot\gamma = (\nabla_{\dot\gamma} X)^{\perp}$

<u>Proof of the theorem</u>

Start from
$$\frac{d}{ds} l(\varphi_s) = \int_a^b \frac{\langle \nabla_T X, T \rangle}{\|T\|} \, dt$$

Note:

$$\frac{\partial}{\partial s}\left(\frac{\langle \nabla_T X, T\rangle}{\|T\|} \right) = \frac{\frac{\partial}{\partial s}\langle \nabla_T X, T\rangle}{\|T\|} - \langle \nabla_T X, T\rangle \cdot \frac{\frac{\partial}{\partial s}\overset{\sqrt{\langle T,T\rangle}}{\|T\|}}{\|T\|^2}$$

$$= \frac{\langle \nabla_X \nabla_T X, T\rangle + \langle \nabla_T X, \nabla_X T\rangle}{\|T\|} - \langle \nabla_T X, T\rangle \frac{\langle \nabla_T X, T\rangle}{\|T\|^3}$$

$$= \frac{\langle \nabla_T \nabla_X X, T\rangle - R(T,X,T,X) + \|\nabla_T X\|^2}{\|T\|} - \frac{\langle \nabla_T X, T\rangle^2}{\|T\|^3}$$

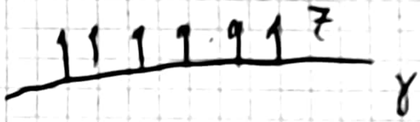
At $s=0$ ($\|T\|=1$, $T=\dot\gamma$, $\nabla_{\dot\gamma}\dot\gamma = 0$):

$$\ldots = \langle \nabla_{\dot\gamma} \nabla_X X, \dot\gamma \rangle - R(\dot\gamma, X, \dot\gamma, X) + \|\nabla_{\dot\gamma} X\|^2 - \langle \nabla_{\dot\gamma} X, \dot\gamma\rangle^2$$

$$= \dot\gamma\left(\langle \nabla_X X, \dot\gamma \rangle\right) - R(\dot\gamma, X, \dot\gamma, X) + \|\nabla_{\dot\gamma} X^{\perp}\|^2$$

Integrating over $t \in [a,b]$: We obtain the required formula $\square$

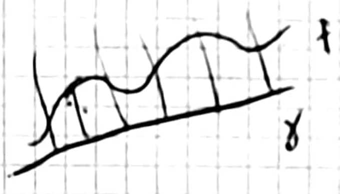
Special case:



Let $\gamma : [0,1] \to M$ be a geodesic with $\|\dot\gamma\| = 1$

and $Z \in \Gamma_\gamma$ with $Z \perp \dot\gamma$, $\nabla_{\dot\gamma} Z = 0$

Let $\phi(t,s) : \exp_{\gamma(t)}(s \cdot f(t) \cdot Z)$



Note: $\nabla_X X = 0$ ($s \to \phi(t,s)$ is geodesic)

then $\frac{d^2}{ds^2} l(\phi_s)\Big|_{s=0} = \int_0^1 \left(\|\nabla_{\dot\gamma} X^\perp\| - R(\dot\gamma, X, \dot\gamma, X)\right) dt$

But: $X = f(t) \cdot Z$

$\Rightarrow \nabla_{\dot\gamma} X = f' \cdot Z$

So $\frac{d^2}{ds^2} l(\phi_s)\Big|_{s=0} = \int_0^1 \left((f')^2 - K(\dot\gamma, Z) \cdot f^2\right) dt$

So: If f "oscillates" a lot: $l(\phi_s)$ increases

If K is > 0, f almost flat: $l(\phi_s)$ decreases

(compare with the case of an equatorial circle in S^2)

Theorem (Bonnet - Myers)

Let (M^n, g) be a complete and connected Riem. manifold with Ric tensor satisfying $Ric \geq a^2(n-1) > 0$, i.e. $\forall X$

$$Ric(X,X) \geq a^2(n-1)\|X\|^2$$

Then $diam(M,g) \leq \frac{\pi}{a}$ (and M is compact).

Rmk: The extreme value is attained for the sphere of radius $R = \frac{1}{a}$.

Rmk: Let $\tilde{M}$ be the universal cover of M $\Rightarrow$ $(\tilde{M}, g)$ also satisfies the above properties $\Rightarrow$ $\tilde{M}$ is compact

$\Rightarrow$ $\pi_1(M)$ is finite

(So: No metric with strictly positive Ricci tensor on T^n)

Rmk: Positivity of the Ricci tensor implies topological constraints on M

But: Every M ($\dim M \geq 3$) admits metric with negative Ricci tensor (Lohkamp)